PERIODIC TRAJECTORIES FOR HOMEOMORPHISMS OF THE CIRCLE WITH BREAK-TYPE SINGULARITIES

D.V. Khmelev

University of Toronto

Certain theorems on a.e. periodicity of one-parameter families of homeomorphisms of the circle with singularities of break type (jump in derivative) are proven. They find application to the following dynamical system suggested by V.I. Arnold. Consider a convex closed set S. Suppose that its boundary ∂S is a curve of length 1. An *involution* I_a of this curve with respect to the line (*direction*) a is defined as follows. Consider all translations of line a. Since the set S is convex, any translation of a meets ∂S at none, one or two points. In the last case we exchange these two points. For any two directions a and b the convolution $\mathrm{I}_b \circ \mathrm{I}_a$ of corresponding involutions defines an orientation-preserving homeomorphism of the boundary ∂S. Suppose that ∂S is smooth everywhere except at one point C, where we have a corner (see Figure 1). Let us also choose a and b passing through the point C.

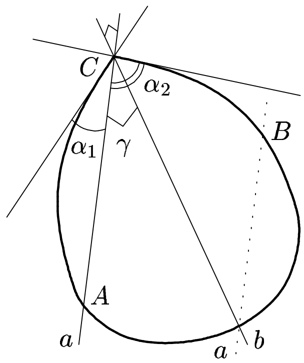

Figure 1: $\mathrm{I}_b \circ \mathrm{I}_a$

Suppose now that ∂S is parameterized by the *arc length parameter*: $x = x(l)$, $y = y(l)$, where $l \in [0, 1)$, so that 0 corresponds to a singular point C. Suppose that one of the following assumptions holds: i) $x(l), y(l) \in C^3([0, 1])$ and all tangent lines to ∂S have non-degenerous tangency, ii) $x(l), y(l) \in C^\infty([0, 1])$ and there are no flat points.

Also assume that the curve $(x(l), y(l))$ has finite one-sided curvatures which are $k_+ > 0$ at $l = 0$ and $k_- > 0$ at $l = 1$. Notice that the last condition is obviously satisfied in the case i). The number of (break-type) singularities for the homeomorphism $\mathrm{I}_b\,\mathrm{I}_a$ obtained depends on the choice of a and b.

Theorem. Directions a and b are defined by angles ζ_1 an $\zeta_2 \in [0, \pi)$ with two tangent lines at point C resp. The Lebesgues measure of the set

$$Z = \{(\zeta_1, \zeta_2) \in [0, \pi) \times [0, \pi) \mid \mathrm{I}_b \circ \mathrm{I}_a \text{ has no periodic trajectories}\} \text{ is } 0.$$

This theorem is a corollary of a more general result on one-parameter families of homeomorphisms of the circle with two break-type singularities. Similar result holds for several breaks if one break is "larger" in a sense than the others.

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada, M5S 3G3
E-mail: dkhmelev@math.toronto.edu